Beginning on page 53

$$\frac{i}{2V}\begin{pmatrix} \underbrace{\left(a_{\mathbf{k}}a_{\mathbf{k}}^{\dagger}-a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}\right)}_{\text{must }=1}\underbrace{e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}})t}}_{=1}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+\underbrace{\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)}_{\text{must }=1}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \quad (\leftarrow \mathbf{k}'=\mathbf{k}) \\ +\underbrace{\left(a_{\mathbf{k}}a_{-\mathbf{k}}^{\dagger}-a_{-\mathbf{k}}^{\dagger}a_{\mathbf{k}}\right)}_{\text{must }=0}e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}+\underbrace{\left(b_{-\mathbf{k}}b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}^{\dagger}b_{-\mathbf{k}}\right)}_{\text{must }=0}e^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})} \quad (\leftarrow \mathbf{k}'=-\mathbf{k}) \end{pmatrix}=\frac{i}{2V}\begin{pmatrix} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \\ +e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\end{pmatrix}$$

Remaining terms where $\mathbf{k}' = \pm\,\mathbf{k}$, *i.e., those of form* $exp(i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}})t)$

Key commutators must = 1

$\left(\text{All time independent terms in summation with } \mathbf{k}' = \pm\,\mathbf{k} \text{ must equal RHS}\right).$

(3-47)

All terms with ($\mathbf{x} + \mathbf{y}$) in the exponents of the LHS must equal zero, as the RHS only has terms in ($\mathbf{x} - \mathbf{y}$). ~~The only way~~ The LHS of (3-47) matches the RHS ~~is~~ if each coefficient commutator in the first row equals one. Subtleties in justifying that as the only way to interpret (3-47) are shown in Appendix E.

The commutation relations for $a_{\mathbf{k}}a_{\mathbf{k}'}{}^{\dagger}$ and $b_{\mathbf{k}}b_{\mathbf{k}'}{}^{\dagger}$ in (3-45) to (3-47) are the same as (3-41). QED.

If you are ambitious, have extra time, and/or simply have to prove everything to yourself, do Prob. 7 to derive the continuous solution commutators of (3-41).

<u>End of coefficient commutation relations proof</u>

Appendix E: Justifying (3-47) Conclusions

Note that (3-47) is one term in a sum over $\mathbf{k}$, where for each term in $\mathbf{k}$ there is an additional one in $-\,\mathbf{k}$. Writing out two such terms leads to

$$\begin{gathered}\left[a_{\mathbf{k}},a_{\mathbf{k}}^{\dagger}\right]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+\left[b_{\mathbf{k}},b_{\mathbf{k}}^{\dagger}\right]e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+\left[a_{\mathbf{k}},a_{-\mathbf{k}}^{\dagger}\right]e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}+\left[b_{-\mathbf{k}},b_{\mathbf{k}}^{\dagger}\right]e^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}\\ \left[a_{-\mathbf{k}},a_{-\mathbf{k}}^{\dagger}\right]e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+\left[b_{-\mathbf{k}},b_{-\mathbf{k}}^{\dagger}\right]e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+\left[a_{-\mathbf{k}},a_{\mathbf{k}}^{\dagger}\right]e^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}+\left[b_{\mathbf{k}},b_{-\mathbf{k}}^{\dagger}\right]e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}\\ =e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+e^{i-\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}=2e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+2e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},\end{gathered}\tag{1}$$

and thus,

$$\begin{aligned}\left[a_{\mathbf{k}},a_{\mathbf{k}}^{\dagger}\right]+\left[b_{-\mathbf{k}},b_{-\mathbf{k}}^{\dagger}\right]&=2 \qquad & \left[b_{\mathbf{k}},b_{\mathbf{k}}^{\dagger}\right]+\left[a_{-\mathbf{k}},a_{-\mathbf{k}}^{\dagger}\right]&=2\\ \left[a_{\mathbf{k}},a_{-\mathbf{k}}^{\dagger}\right]+\left[b_{\mathbf{k}},b_{-\mathbf{k}}^{\dagger}\right]&=0 & \left[b_{-\mathbf{k}},b_{\mathbf{k}}^{\dagger}\right]+\left[a_{-\mathbf{k}},a_{\mathbf{k}}^{\dagger}\right]&=0\end{aligned}\tag{2}$$

At this point, we could adopt a reasonable postulate that coefficients in (2) not having the same 3-momentum have zero commutators, and those that do have the same 3-momentum all have the same commutator values. That would give us (3-41) and lead to our present day (good) theory of QFT.

If we were to be thorough, however, and repeat the process of (3-42) to (3-47) for other commutators, such as $\left[\phi,\phi^{\dagger}\right]=0$, we would find other relations between coefficient commutators that would lead inevitably to (3-41). You can take my word for this, work it out yourself (which is tedious), or see it on the book website under Auxiliary Material (URL on pg. xvi, opposite pg. 1).

<u>End of change to book.</u> Material below found on book website.

Working It Out in Detail

Instead of (3-42), use one of the other relations in (3-40), where $\phi^{r}=\phi,\ \phi^{s}=\phi^{\dagger}$,

$$\left[\phi,\phi^{\dagger}\right]=0\,.\tag{3}$$

So instead of (3-43), we have

$$\sum_{\mathbf{k}'}\frac{1}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\begin{pmatrix} a_{\mathbf{k}}b_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}+a_{\mathbf{k}}a_{\mathbf{k}'}^{\dagger}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\\ b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}+b_{\mathbf{k}}^{\dagger}a_{\mathbf{k}'}^{\dagger}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}\\ -b_{\mathbf{k}'}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}-b_{\mathbf{k}'}b_{\mathbf{k}}^{\dagger}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\\ -a_{\mathbf{k}'}^{\dagger}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}-a_{\mathbf{k}'}^{\dagger}b_{\mathbf{k}}^{\dagger}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}\end{pmatrix}=0\,.\tag{4}$$

Instead of (3-45), we get

$$
\begin{aligned}
&\underbrace{\left(a_{\mathbf{k}} b_{\mathbf{k}'}-b_{\mathbf{k}'} a_{\mathbf{k}}\right)}_{\text{must}=0} e^{-i\left(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}\right) t} e^{i\left(\mathbf{k} \cdot \mathbf{x}+\mathbf{k}' \cdot \mathbf{y}\right)}+\underbrace{\left(a_{\mathbf{k}} a_{\mathbf{k}'}^{\dagger}-a_{\mathbf{k}'}^{\dagger} a_{\mathbf{k}}\right)}_{\text{must}=0} e^{-i\left(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}\right) t} e^{i\left(\mathbf{k} \cdot \mathbf{x}-\mathbf{k}' \cdot \mathbf{y}\right)} \\
&+\underbrace{\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}'}-b_{\mathbf{k}'} b_{\mathbf{k}}^{\dagger}\right)}_{\text{must}=0} e^{i\left(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}\right) t} e^{-i\left(\mathbf{k} \cdot \mathbf{x}-\mathbf{k}' \cdot \mathbf{y}\right)}+\underbrace{\left(b_{\mathbf{k}}^{\dagger} a_{\mathbf{k}'}^{\dagger}-a_{\mathbf{k}'}^{\dagger} b_{\mathbf{k}}^{\dagger}\right)}_{\text{must}=0} e^{i\left(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}\right) t} e^{-i\left(\mathbf{k} \cdot \mathbf{x}+\mathbf{k}' \cdot \mathbf{y}\right)}=0,
\end{aligned}
\tag{5}
$$

which gives us the same coefficient commutators as with (3-45). Note that had we included the – **k** part of the summation in (5) we would not have had terms from that with exponents matching any terms in (5) shown above. (For example the first term in (5) with **k** → – **k**, becomes $\left(a_{-\mathbf{k}} b_{\mathbf{k}'}-b_{\mathbf{k}'} a_{-\mathbf{k}}\right) e^{-i\left(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}\right) t} e^{i\left(-\mathbf{k} \cdot \mathbf{x}+\mathbf{k}' \cdot \mathbf{y}\right)}$, the exponent of which matches none of the terms in (5)) Bottom line: We don't have to worry about the – **k** part of the summation in drawing our conclusions (see underbrackets) in (5).

Instead of (3-46), we get

$$
\begin{aligned}
&\underbrace{\left(a_{\mathbf{k}} b_{\mathbf{k}}-b_{\mathbf{k}} a_{\mathbf{k}}\right)}_{\text{must}=0} \underbrace{e^{-i 2 \omega_{\mathbf{k}} t}}_{\neq 0} e^{i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}+\underbrace{\left(b_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}^{\dagger}-a_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}^{\dagger}\right)}_{\text{must}=0} e^{i 2 \omega_{\mathbf{k}} t} e^{-i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})} \\
&+\underbrace{\left(a_{\mathbf{k}} b_{-\mathbf{k}}-b_{-\mathbf{k}} a_{\mathbf{k}}\right)}_{\text{must}=0} e^{-i 2 \omega_{\mathbf{k}} t} e^{i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}+\underbrace{\left(b_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}^{\dagger}-a_{-\mathbf{k}}^{\dagger} b_{\mathbf{k}}^{\dagger}\right)}_{\text{must}=0} e^{i 2 \omega_{\mathbf{k}} t} e^{-i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}=0,
\end{aligned}
\tag{6}
$$

which gives us the same results as (3-46). Similar reasoning as used to ignore the –**k** terms in the summation in (5) apply to (6).

Instead of (3-47), we get (including the – **k** as we did in (1) above)

$$
\begin{aligned}
&\left(a_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}-a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}\right) e^{i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}+\left(b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}}-b_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right) e^{-i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})} \\
&\qquad+\left(a_{\mathbf{k}} a_{-\mathbf{k}}^{\dagger}-a_{-\mathbf{k}}^{\dagger} a_{\mathbf{k}}\right) e^{i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}+\left(b_{\mathbf{k}}^{\dagger} b_{-\mathbf{k}}-b_{-\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right) e^{-i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})} \\
&\left(a_{-\mathbf{k}} a_{-\mathbf{k}}^{\dagger}-a_{-\mathbf{k}}^{\dagger} a_{-\mathbf{k}}\right) e^{-i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}+\left(b_{-\mathbf{k}}^{\dagger} b_{-\mathbf{k}}-b_{-\mathbf{k}} b_{-\mathbf{k}}^{\dagger}\right) e^{i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})} \\
&\qquad+\left(a_{-\mathbf{k}} a_{\mathbf{k}}^{\dagger}-a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}\right) e^{-i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}+\left(b_{-\mathbf{k}}^{\dagger} b_{\mathbf{k}}-b_{\mathbf{k}} b_{-\mathbf{k}}^{\dagger}\right) e^{i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}=0.
\end{aligned}
\tag{7}
$$

Or re-written

$$
\begin{aligned}
&\left[a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\right] e^{i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}-\left[b_{\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right] e^{-i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}+\left[a_{\mathbf{k}}, a_{-\mathbf{k}}^{\dagger}\right] e^{i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}-\left[b_{-\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right] e^{-i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})} \\
&\left[a_{-\mathbf{k}}, a_{-\mathbf{k}}^{\dagger}\right] e^{-i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}-\left[b_{-\mathbf{k}}, b_{-\mathbf{k}}^{\dagger}\right] e^{i \mathbf{k} \cdot(\mathbf{x}-\mathbf{y})}+\left[a_{-\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\right] e^{-i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}-\left[b_{\mathbf{k}}, b_{-\mathbf{k}}^{\dagger}\right] e^{i \mathbf{k} \cdot(\mathbf{x}+\mathbf{y})}=0.
\end{aligned}
\tag{8}
$$

From (8) we have

$$
\begin{array}{ll}
\left[a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\right]=\left[b_{-\mathbf{k}}, b_{-\mathbf{k}}^{\dagger}\right] & \left[b_{\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right]=\left[a_{-\mathbf{k}}, a_{-\mathbf{k}}^{\dagger}\right] \\
\left[a_{\mathbf{k}}, a_{-\mathbf{k}}^{\dagger}\right]=\left[b_{\mathbf{k}}, b_{-\mathbf{k}}^{\dagger}\right] & \left[b_{-\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right]=\left[a_{-\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\right]
\end{array}
\tag{9}
$$

Using (9) in (2), we find

$$
\begin{array}{ll}
\left[a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\right]=\left[b_{-\mathbf{k}}, b_{-\mathbf{k}}^{\dagger}\right]=1 & \left[b_{\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right]=\left[a_{-\mathbf{k}}, a_{-\mathbf{k}}^{\dagger}\right]=1 \\
\left[a_{\mathbf{k}}, a_{-\mathbf{k}}^{\dagger}\right]=\left[b_{\mathbf{k}}, b_{-\mathbf{k}}^{\dagger}\right]=0 & \left[b_{-\mathbf{k}}, b_{\mathbf{k}}^{\dagger}\right]=\left[a_{-\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\right]=0,
\end{array}
\tag{10}
$$

and thus with (5), (6), and (10), we get (3-41).